Generalized Frames in Finite Space and Applications

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Abstract. In the field of signal and imaging processing, the Frame Theory is a practical-mathematical tool for analysis. Moreover, generalized frames such as K-frame and g-frame are also hot research topics in the field of analytics. We apply the conclusions of the frames in infinite-dimensional to the finite-dimensional space, which makes it easier for us to apply the conclusions in actual scenarios.

Keywords: finite dimensional frames; generalized frames; dynamical sampling.

1. Introduction

The fundamental property of frames is recovering signals from additive noise. In 1946, D. Gabor introduced a new signal decomposition method, which begins the research on the frame theory. In 1952, Duffin and Schaeffer formally introduced the notion of frames in Hilbert space when studying non-harmonic Fourier analysis. In 1986, Daubechies, Grossman and Mayer reintroduced the basic concept and properties of frames, and they showed the importance of frame for data processing.

In the 21st century, frame theory has developed rapidly, and many generalized frames have appeared. Among them, the most famous generalized frames are K-frame and g-frame. The notion of K-frame was first proposed in \cite{6}. The notion of g-frame was first proposed in \cite{10}. Now, these two frames are still being studied by many mathematicians in the field of analysis. Of course, research is not limited to the properties of the two frames themselves. \cite{7} details the notion of g-atomic subspace and constructs several useful resolutions of the identity operator on a Hilbert space using the theory of g-fusion frames. \cite{2} revealed the important properties of Bessel K-fusion sequences and of controlled K-fusion frames. These two frames are still attracting attention today.

In addition, many mathematicians are studying the specific applications of frames. This is closely related to finite-dimensional frames. It plays an outstanding role in signal processing, image processing, data compression, sampling theory, abstract mathematics, seismic exploration, geophysics, radar and communications, and has very broad application prospects. However, few people consider finite-dimensional generalized frames, therefore, we are going to study the properties of finite-dimensional K-frames and g-frames. There are also many potential applications in finite-dimensional frames.

The main text of this article is going to be divided into the following three parts: The first part, in section 2, is the finite dimensional frame. We would introduce some definitions and properties or theorems that have been proved by previous people. Then, in section 3 and section 4, we will introduce the definitions and properties of finite-dimensional K-frame and g-frame. The final part is the application part. We would introduce how to construct a frame and take dynamic sampling as an example to introduce how finite-dimensional K-frame and g-frame are applied in practical problems. Then, we would end with an introduction to the Gabor frame.

In this article, we denote by $\mathcal{H}^n$ an n-dimensional complex separable Hilbert space, with inner product $\langle \cdot, \cdot \rangle$, and norm $\| \cdot \|$. Let $\mathcal{H}_1, \mathcal{H}_2$ be two Hilbert spaces. We denote by $\mathcal{L}(\mathcal{H}_1, \mathcal{H}_2)$ the space of all linear bounded operators from $\mathcal{H}_1$ to $\mathcal{H}_2$. If $\mathcal{H}_1 = \mathcal{H}_2 = \mathcal{H}$, we call $\mathcal{L}(\mathcal{H})$ the space of all linear bounded operators from $\mathcal{H}$ to $\mathcal{H}$. In particular, we denote by $\text{Id}$ the identity operator in $\mathcal{L}(\mathcal{H})$. We denote by $R(\cdot)$ the range of an operator. We denote by $l^2(m)$ m-dimensional Banach space with norm

$$
\|(x_1, x_2, \cdots, x_m)\|_2 = \left( \sum_{i=1}^{m} x_i^2 \right)^{\frac{1}{2}}, \quad \forall (x_1, x_2, \cdots, x_m) \in l^2(m).
$$
2. Finite Dimensional Frames

We begin with introducing the definitions of the frame.

**Definition 2.1** A sequence \( \{f_i\}_{i=1}^m \subseteq \mathcal{H}^n \) is called a frame for \( \mathcal{H}^n \) if there exist \( A, B > 0 \), such that

\[
A\|f\|^2 \leq \sum_{k=1}^{m} |(f, f_k)|^2 \leq B \|f\|^2, \quad \forall f \in \mathcal{H}^n.
\]

**Remark:** Here \( A \) is called the upper bound of the frame and \( B \) is the lower bound. If \( A = B \), we call \( \{f_i\}_{i=1}^m \) a tight frame. If \( A = B = 1 \), we call \( \{f_i\}_{i=1}^m \) a Parseval frame. If just the last inequality holds, we say that \( \{f_i\}_{i=1}^m \) is a Bessel sequence.

In the space of probabilities, there is also a type of frame. That is the following.

**Definition 2.2** Let \( \Omega \) be a subset of \( \mathcal{H}^n \). A probability measure \( \mu \) is called a probabilistic frame for \( \mathcal{H}^n \) if there exist \( A, B > 0 \), such that

\[
A\|f\|^2 \leq \int_{\Omega} |(f, g)|^2 d\mu(g) \leq B \|f\|^2, \quad \forall f \in \mathcal{H}^n.
\]

**Definition 2.3** A sequence \( \{f_i\}_{i=1}^m \subseteq \mathcal{H}^n \) is called a Riesz basis for \( \mathcal{H}^n \) if \( \mathcal{H}^n = \text{span}\{f_i\}_{i=1}^m \) and there exist \( A, B > 0 \), such that for every \( a \in K, 1 \leq i \leq m \), we have

\[
A \sum_{k=1}^{i} |a_k|^2 \leq \left\| \sum_{k=1}^{i} a_k f_{nk} \right\|^2 \leq B \sum_{k=1}^{i} |a_k|^2
\]

holds for every \( \{f_{nk}\}_{k=1}^i \).

We would like to introduce the following operators three operators to study the frame theory.

**Definition 2.4** Let \( \mathcal{H}^n \) be an \( n \)-dimensional Hilbert space, \( \{f_i\}_{i=1}^m \subseteq \mathcal{H}^n \). We say \( T \) is associated analysis operator with \( \{f_i\}_{i=1}^m \), if \( T: \mathcal{H}^n \rightarrow l^2(m) \) such that

\[
T_x = ((x, f_i))_{i=1}^m, \quad \forall x \in \mathcal{H}^n.
\]

**Lemma 2.1** Suppose a sequence \( \{f_i\}_{i=1}^m \subseteq \mathcal{H}^n \). Let \( T: \mathcal{H}^n \rightarrow l^2(m) \) be associated analysis operator with \( \{f_i\}_{i=1}^m \). Then \( \{f_i\}_{i=1}^m \) is a frame if and only if \( T \) is injective.

**Definition 2.5** Let \( \mathcal{H}^n \) be a Hilbert space, \( \{f_i\}_{i=1}^m \subseteq \mathcal{H}^n \). \( T^* \) is associated synthesis operator with \( \{f_i\}_{i=1}^m \). Then, the associated synthesis operator with \( \{f_i\}_{i=1}^m \) is defined by

\[
T^*: l^2(m) \rightarrow \mathcal{H}^n, \quad (a_i)_{i=1}^m \mapsto \sum_{i=1}^{m} a_i f_i, \quad \forall (a_i)_{i=1}^m \in l^2(m).
\]

**Lemma 2.2** Suppose a sequence \( \{f_i\}_{i=1}^m \subseteq \mathcal{H}^n \). Let \( T^*: l^2(m) \rightarrow \mathcal{H}^n \) be associated synthesis operator with \( \{f_i\}_{i=1}^m \). Then \( \{f_i\}_{i=1}^m \) is a frame if and only if \( T^* \) is surjective.

By the following frame operator defined by the above \( T, T^* \), we could investigate frames easier.

**Definition 2.6** Let \( \mathcal{H}^n \) be a Hilbert space, \( \{f_i\}_{i=1}^m \subseteq \mathcal{H}^n \). Let \( T: \mathcal{H}^n \rightarrow l^2(m) \) be associated analysis operator with \( \{f_i\}_{i=1}^m \). Then, the frame operator \( S \) is defined as follows:

\[
S: \mathcal{H}^n \rightarrow \mathcal{H}^n, \quad Sf = \sum_{j=1}^{m} (f, f_j) f_j, \quad \forall f \in \mathcal{H}^n.
\]

In a finite-dimensional space, the matrix form of an operator is very important. We would introduce it below.

**Definition 2.7** Let \( T: \mathcal{H}^n \rightarrow \mathcal{H}^m \) be a linear operator. \( \{e_i\}_{i=1}^n \) is an orthonormal basis of \( \mathcal{H}^n \) and \( \{f_i\}_{i=1}^m \) is an orthonormal basis of \( \mathcal{H}^m \). Then, the matrix representation of \( T \) under \( \{e_i\}_{i=1}^n \) and \( \{f_i\}_{i=1}^m \) is

\[
T = \begin{bmatrix}
(Te_1, f_1) & (Te_2, f_1) & \cdots & (Te_n, f_1) \\
(Te_1, f_2) & (Te_2, f_2) & \cdots & (Te_n, f_2) \\
\vdots & \vdots & \ddots & \vdots \\
(Te_1, f_m) & (Te_2, f_m) & \cdots & (Te_n, f_m)
\end{bmatrix}
\]
∀𝑥 ∈ ℋⁿ, 𝑇𝑥 = ̂𝑇̂𝑥 , where ̂𝑥 = ((𝑥, 𝑒₁), 𝑥, 𝑒₂), · · · , (𝑥, 𝑒ₙ)).

3. Finite Dimensional 𝑆-frames and Its Propositions

In this section, we would discuss the first generalized frame. We first introduce some definitions.

Definition 3.1 Let 𝑆 ∈ ℒ(ℋⁿ). A sequence \{𝑓ᵢ\}ᵢ=1ⁿ ∈ ℋⁿ is called a 𝑆-frame for ℋⁿ, if there exist 𝐴, 𝐵 > 0, such that

\[ A\|𝑆∗𝑓\|² ≤ \sum_{k=1}^{m} |⟨𝑓, 𝑓_k⟩|^² ≤ B \|𝑓\|², \quad ∀ 𝑓 ∈ ℋⁿ. \]

Remark: When 𝑆 = 𝐼, then \{𝑓ᵢ\}ᵢ=1ⁿ is a 𝑆-frame if and only if \{𝑓ᵢ\}ᵢ=1ⁿ is a frame. Thus, we call 𝑆-frame a generalized frame.

Then, we would like to find necessary and sufficient conditions that the sequence \{𝑓ᵢ\}ᵢ=1ⁿ is a 𝑆-frame. Thus, we need the following lemma.

Lemma 3.1 Let 𝑇₁ ∈ ℒ(ℋ₁, ℋ₂), 𝑇₂ ∈ ℒ(ℋ₂, ℋ₃) be two operators. The following statements are equivalent:

\[ R(𝑇₁) ⊂ R(𝑇₂); \]
\[ 𝑇₁𝑇₁^* ≤ λ^2 𝑇₂𝑇₂^* \text{ for some } 𝜆 ≥ 0; \]

There exists \( X ∈ ℒ(ℋ₁, ℋ₂) \) so that \( 𝑇₁ = 𝑇₂X \).

Now, we would give necessary and sufficient conditions that the sequence \{𝑓ᵢ\}ᵢ=1ⁿ is a 𝑆-frame. Thus, we need the following lemma.

Theorem 3.1 Let ℋⁿ be an 𝑛-dimensional separable Hilbert space and \{𝑒ᵢ\}ᵢ=1ⁿ is an orthonormal basis for \( l²(𝑚) \). Then the followings are equivalent:

\{𝑓ᵢ\}ᵢ=1ⁿ is a 𝑆-frame for ℋⁿ.

\{𝑓ᵢ\}ᵢ=1ⁿ is a Bessel sequence and there exists a Bessel sequence \{𝑔ᵢ\}ᵢ=1ⁿ such that

\[ 𝑆𝑓 = \sum_{i=1}^{m} ⟨𝑓, 𝑔ᵢ⟩𝑓ᵢ. \]

There exist \( C > 0 \) such that \( ∀ 𝑓 ∈ ℋⁿ \), there exists \( 𝑎_𝑓 = (𝑎_𝑖)ᵢ=1ⁿ ∈ l²(𝑚) \) such that \( \|𝑎_𝑓\|₂ ≤ C\|𝑓\| \) and \( 𝑆𝑓 = \sum_{k=1}^{m} 𝑎_𝑘𝑓_𝑘 \).

There exists \( L ∈ ℒ(l²(𝑚), ℋⁿ) \) such that \( 𝑓ᵢ = 𝐿𝑒ᵢ \) and \( R(𝑆) ⊂ R(𝐿) \).

Proof. If we have (1), then \{𝑓ᵢ\}ᵢ=1ⁿ is also a Bessel sequence. Moreover, there exist \( T: l²(𝑚) → ℋⁿ \) such that \( 𝑓ᵢ = 𝑇𝑒ᵢ \), where \( 𝑒ᵢ = (0, · · · , 1, 0, · · · 0) \). Since \{𝑓ᵢ\}ᵢ=1ⁿ is a 𝑆-frame, \( T \) is linear. Then,

\[ A\|𝑆∗𝑓\|² ≤ \sum_{k=1}^{m} |⟨𝑓, 𝑓_k⟩|^² = \sum_{k=1}^{m} |⟨𝑓, 𝑇𝑒ₖ⟩|^² = \sum_{k=1}^{m} |⟨𝑇^∗𝑓, 𝑒ₖ⟩|^² = \|𝑇^∗𝑓\|². \]

From the Lemma 3.1, there exists a linear operator \( M: ℋⁿ → l²(𝑚) \) such that \( 𝑆 = TM \). Define \( F_k: ℋ → ℂ, x ↦ ⟨𝑀𝑓, 𝑒_k⟩ = 𝑎_𝑘 \). Denote \( a = 𝑀𝑓 = (𝑎_𝑖)ᵢ=1ⁿ \), we have

\[ |𝑎_𝑘| ≤ \left( \sum_{k=1}^{m} |𝑎_𝑘|^² \right)^{1/2} = \|a\|₂ ≤ \|𝑀\| \|𝑓\|. \]

By the Riesz representation theorem, there exists \( gₙ ∈ ℋⁿ \) such that \( 𝑎_𝑘 = ⟨𝑥, 𝑔ₙ⟩ \). Then,

\[ 𝑆𝑓 = 𝑇𝑀𝑓 = 𝑇(𝑎) = \sum_{k=1}^{m} 𝑎_𝑘𝑓_𝑘 = \sum_{k=1}^{m} ⟨𝑓, 𝑔_𝑘⟩𝑓_𝑘. \]

Moreover,

\[ \sum_{i=1}^{m} |⟨𝑓, 𝑔ᵢ⟩|^² = \sum_{i=1}^{m} 𝑎_𝑖² ≤ \|𝑀\|² \|𝑓\|². \]

It implies that \{𝑔ᵢ\}ᵢ=1ⁿ is a Bessel sequence.

If we have (2), we take \( 𝑎_𝑖 = ⟨𝑓, 𝑔ᵢ⟩ \) and \( C = \|𝑀\|² \). We directly have (3).
If we have (3), we have 
\[ \|K^*f\| = \sup_{\|g\|=1} |\langle K^*f, g \rangle| = \sup_{\|g\|=1} |\langle f, Kg \rangle| . \]

Let \( K \) defined as 
\[ K = \sum_{i=1}^{m} b_i f_i , \]
we would obtain that
\[ \|K^*f\| = \sum_{i=1}^{m} |\langle x, f_i \rangle| \leq \sum_{i=1}^{m} |\langle x, b_i \rangle| \leq C_0 \left( \sum_{i=1}^{m} |\langle x, f_i \rangle|^2 \right)^{\frac{1}{2}} . \]

where \( C_0 \) is the norm of the operator \( K \). Thus, we obtain
\[ \frac{1}{C_0^2} \|K^*f\|^2 \leq \sum_{i=1}^{m} |\langle x, f_i \rangle|^2 \leq C^2 \|f\|^2 . \]

Therefore, we have (1), (2), (3) are equivalent.

Then we prove (1) is equivalent to (4). We first prove (1) can implies (4). Let \( T: \mathcal{H} \rightarrow l^2 \) \( f \mapsto \sum_{k=1}^{m} (f, f_k)e_k \) Since
\[ \langle T^* e_k, x \rangle = \langle e_k, T x \rangle = \sum_{k=1}^{m} \langle x, f_k \rangle e_k = \sum_{k=1}^{m} \langle x, f_k \rangle (e_k, e_i) = \langle x, f_k \rangle = \langle f_k, x \rangle . \]

we have \( T^* e_k = f_k \) for all \( k = 1,2,\ldots, m \). Moreover, \( \{f_i\}_{i=1}^{m} \) is a \( K \)-frame for \( \mathcal{H} \), there exist \( A, B > 0 \), such that
\[ A \|K^*f\|^2 \leq \sum_{k=1}^{m} |\langle f, f_k \rangle|^2 \leq B \|f\|^2 , \quad \forall f \in \mathcal{H} . \]

Therefore, we have \( A \|K^*f\|^2 \leq \|T(x)\|^2 \). Take \( L = T^* \). We have \( AK^* \leq TT^* \). By Lemma 3.1, we have \( R(K) \subset R(L) \).

Next, we prove (4) can implies (1). From \( f_i = Le_i \), we have
\[ \langle L^* f, g \rangle = \left( L^* f, \sum_{k=1}^{m} a_k e_k \right) = \sum_{k=1}^{m} \overline{a_k} \langle L^* f, e_k \rangle = \sum_{k=1}^{m} \overline{a_k} \langle f, Le_k \rangle = \sum_{k=1}^{m} \overline{a_k} \langle g, e_k \rangle \langle f, f_k \rangle = \sum_{k=1}^{m} \langle e_k, g \rangle \langle f, f_k \rangle \]
\[ = \left( \sum_{k=1}^{m} \langle f, f_k \rangle e_k, g \right) , \quad \forall g \in \mathcal{H} . \]

Therefore, \( L^* f = \sum_{k=1}^{m} \langle f, f_k \rangle e_k \). Then
\[ \sum_{k=1}^{m} |\langle f, f_k \rangle|^2 = \|L^* f\|^2 \leq \|L^*\|^2 \|f\|^2 . \]

Moreover \( R(K) \subset R(L) \), From Lemma 3.1, we have \( \exists \alpha > 0 \) such that \( AK^* \leq LL^* \). Hence,
\[ A \|K^*f\|^2 \leq \|L^* f\|^2 = \sum_{k=1}^{m} |\langle f, f_k \rangle|^2 \leq \|L^*\|^2 \|f\|^2 . \]

Thus, (1) is equivalent to (4).

Till now, we have successfully obtained the necessary and sufficient conditions for the ordinary \( K \)-frame. Next, we would study another form of \( K \)-frame, namely probabilistic \( K \)-frame. We introduce the definitions first.

**Definition 3.2** Let \( \mathcal{H} \) be an \( n \)-dimensional separable Hilbert space, \( K \in \mathcal{L} (\mathcal{H}) \) and \( \Omega \) be a bounded subset of \( \mathcal{H} \). Then we have
(1) A probability measure \( \mu \) is called a probabilistic \( K \)-frame for \( \mathcal{H} \) if there exist \( A, B > 0 \), such that
\[ A \|K^*f\|^2 \leq \int_{\Omega} |\langle f, g \rangle|^2 d\mu(g) \leq B \|f\|^2 , \quad \forall f \in \mathcal{H} . \]
The support of \( \mu \) denoted by \( \text{supp}(\mu) \) is that \( \text{supp}(\mu) := \{ x \in \Omega : \mu(Ux) > 0 \} \) for all open subsets \( Ux \subseteq K \) that contain \( x \).

Similarly, we would like to give a necessary and sufficient condition that \( \mu \) is a probabilistic \( K \)-frame. That is the following theorem.

**Theorem 3.2** Let \( \mathcal{H}^n \) be an \( n \)-dimensional separable Hilbert space. Let \( \Omega \) be a bounded subset of \( \mathcal{H}^n \). A probability measure \( \mu \) is a probabilistic \( K \)-frame if and only if its support spans \( \mathcal{H}^n \).

**Proof.** Let \( \text{span}(\text{supp}(\mu)) = (\text{supp}(\mu)) \). If \( \langle \text{supp}(\mu) \rangle \neq \mathcal{H}^n \), there exist \( x \in (\text{supp}(\mu))^\perp \) such that \( \int_{\Omega} \langle x, y \rangle^2 d\mu(y) = 0 \), which is contradict to the fact that \( \mu \) is a probabilistic \( K \)-frame.

On the other hand, by Cauchy-Schwartz inequality, we have

\[
\int_{\Omega} |\langle f, g \rangle|^2 d\mu(g) \leq \sup_{g \in \Omega} (\|g\|) \|f\|, \quad \forall f \in \mathcal{H}^n.
\]

Since \( \Omega \) is bounded, \( \sup_{g \in \Omega} (\|g\|) < \infty \). Therefore, we take \( B = \sup_{g \in \Omega} (\|g\|) \). Moreover, define

\[
A := \inf_{f \in \mathcal{H}^n} \left( \frac{\int_{\Omega} |\langle f, g \rangle|^2 d\mu(g)}{\|K^*\| \|f\|} \right) = \inf_{\|g\| = 1} \left( \frac{1}{\|K^*\|} \int_{\Omega} |\langle f, g \rangle|^2 d\mu(g) \right).
\]

By dominated convergence theorem, the mapping \( \Phi: f \mapsto \int_{\Omega} |\langle f, g \rangle|^2 d\mu(g) \) is continuous. Since the set \( \{ f \in \mathcal{H}^n : \|f\| = 1 \} \) is a bounded closed set, it is compact in \( \mathcal{H}^n \). \( \exists f_0 \in \mathcal{H}^n, \|f_0\| = 1 \), such that

\[
A = \frac{1}{\|K^*\|} \int_{\Omega} |\langle f_0, g \rangle|^2 d\mu(g).
\]

If \( \langle \text{supp}(\mu) \rangle = \mathcal{H}^n \), \( \exists g_0 \in \text{supp}(\mu) \) such that \( |\langle f_0, g_0 \rangle| > 0 \). Thus, \( \exists \varepsilon > 0, U_{g_0} \subseteq \Omega \), such that \( U_{g_0} \) open in \( \Omega \), \( \mu(U_{g_0}) > 0, g_0 \in U_{g_0} \) and \( |\langle f_0, g \rangle| > 0 \), \( \forall g \in U_{g_0} \). Therefore, we have \( A \geq \varepsilon \mu(U_{g_0}) > 0 \). Recall that

\[
A = \inf_{f \in \mathcal{H}^n} \left( \frac{\int_{\Omega} |\langle f, g \rangle|^2 d\mu(g)}{\|K^*\| \|f\|} \right), \quad \text{we have}
\]

\[
A \|K^*f\| \leq A \|K^*\| \|f\| \leq \int_{\Omega} |\langle f_0, g \rangle|^2 d\mu(g).
\]

4. **Finite Dimensional g-Frames and Its Propositions**

In this section, we would like to introduce another generalized frame. Let \( \mathcal{H}^n \) be an \( n \)-dimensional Hilbert space. \( \{ \mathcal{H}_j \}_{j=1}^m \) are subspaces of \( \mathcal{H}^n \) and satisfy

\[
\mathcal{H}^n = \bigoplus_{j=1}^m \mathcal{H}_j.
\]

**Definition 4.1** Let \( \Lambda_j \in \mathcal{L}(\mathcal{H}^n, \mathcal{H}_j) \). We call \( \{ \Lambda_j \}_{j=1}^m \) a generalized frame for \( \mathcal{H}^n \) with respect to \( \{ \mathcal{H}_j \}_{j=1}^m \) if there are \( A, B > 0 \) such that

\[
A \|f\|^2 \leq \sum_{j=1}^m \|\Lambda_j f\|^2 \leq B \|f\|^2, \quad \forall f \in \mathcal{H}^n.
\]

If the right-hand inequality holds, then we say \( \{ \Lambda_j \}_{j=1}^m \) is a g-Bessel sequence for \( \mathcal{H}^n \) with respect to \( \{ \mathcal{H}_j \}_{j=1}^m \).

**Lemma 4.1** Let \( \mathcal{H}^n \) be a separable Hilbert space. Then, an operator sequence \( \{ \Lambda_{f_j} \}_{j=1}^m \) is a g-frame for \( \mathcal{H}^n \) with respect to \( \mathcal{H}_j = K \) if and only if there exist a frame \( \{ f_j \}_{j=1}^m \) for \( \mathcal{H}^n \) such that

\[
\Lambda_{f_j} f = \langle f, f_j \rangle, \quad \forall f \in \mathcal{H}^n.
\]

**Proof.** Since the necessity is obvious, we only prove the sufficiency. According to the Riesz Representation Theorem, \( \forall \Lambda f_j, \exists f_j \in \mathcal{H}^n \), such that \( \Lambda f_j = \langle f, f_j \rangle, \forall f \in \mathcal{H}^n \). Thus, we have
\[ A\|f\|^2 \leq \sum_{j=1}^{m} \|A_j f\|^2 = \sum_{j=1}^{m} |\langle f, f_j \rangle|^2 \leq B\|f\|^2, \quad \forall f \in \mathcal{H}^n. \]

Therefore, \( \{f_j\}_{j=1}^{m} \) is a frame for \( \mathcal{H}^n \).

From the lemma above, we would see that the g-frame is a generalized frame.

**Definition 4.2** Let \( \{A_j\}_{j=1}^{m} \) be a g-frame for \( \mathcal{H}^n \) with respect to \( \{\mathcal{H}_j\}_{j=1}^{m} \). Then, \( S \) is called the g-frame operator if

\[ S: \mathcal{H}^n \rightarrow \mathcal{H}^n, \quad Sf = \sum_{j=1}^{m} A_j^* A_j f, \quad \forall f \in \mathcal{H}^n. \]

**Lemma 4.2** Let \( \mathcal{H}_j = \mathbb{R}^n, \forall j = 1, 2, \cdots, m. \) Then, \( Sf = \sum_{j=1}^{m} A_j^T A_j f \), where \( A_j \) is the matrix representation for \( A_j \). In particular, if \( \mathcal{H}_j = \mathbb{R}, \forall j = 1, 2, \cdots, m \), there exists a frame \( \{f_j\}_{j=1}^{m} \) for \( \mathcal{H}^n \). \( S \) is the frame operator.

**Proof.** We first prove the case of \( \mathcal{H}_j = \mathbb{R} \), By Lemma 4.1, if \( \mathcal{H}_j = \mathbb{R}, \exists f_j \in \mathcal{H}^n \), such that \( A_j f = (f, f_j) \). Then,

\[ \langle A_j^* x, f \rangle = \langle x, A_j^* f \rangle = \langle x, (f, f_j) \rangle = x(f, f_j) = \langle x f_j, f \rangle \]

Therefore, we have \( A_j^* x = x f_j, \forall x \in \mathbb{R} \). Then

\[ Sf = \sum_{j=1}^{m} A_j^* A_j f = \sum_{j=1}^{m} (A_j f) f_j = \sum_{j=1}^{m} (f, f_j) f_j, \quad \forall f \in \mathcal{H}^n. \]

Thus \( S \) is the frame operator.

For the general case, we repeat the steps above. Let \( \{e_i\}_{i=1}^{n} \) be an orthonormal basis of \( \mathcal{H}^n \), there exist some elements in \( \{e_i\}_{i=1}^{n} \), namely \( \{f_{ij}\}_{i=1}^{n} \), which is a basis of \( \mathcal{H}_j \). there exists a matrix

\[ A_j = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{nj,1} & \cdots & a_{nj,n} \end{pmatrix} = (\alpha_1 \cdots \alpha_n) \]

such that \( A_j x = T_j x \). Take \( x^T = (x_1, \cdots, x_n) \), \( f^T = (f_1, \cdots, f_n) \). Thus,

\[ \langle A_j^* x, f \rangle = \langle x, A_j f \rangle = \sum_{i=1}^{n} x_i (a_{i1} f_1 + \cdots a_{in} f_n) = \sum_{i=1}^{n} (a_{i1} x_1 + \cdots a_{nj,i} x_n) f_i = \langle A_j^T f, f \rangle. \]

Therefore, \( A_j^* x = A_j^T f \). Then we have

\[ Sf = \sum_{j=1}^{m} A_j^* A_j f = \sum_{j=1}^{m} A_j^T A_j f. \]

Here, we also want to introduce the definition of g-Riesz basis, we would introduce the definition of the notion of g-complete.

**Definition 4.3** Let \( \Lambda_j \in \mathcal{L}(\mathcal{H}^n, \mathcal{H}_j), j = 1, 2, \cdots, m. \) The sequence \( \{\Lambda_j\}_{j=1}^{m} \) is g-complete if the set \( \{f \in \mathcal{H}^n: \Lambda_j f = 0, j = 1, 2, \cdots, m\} = \{0\}. \)

**Remark:** Applying the definition of kernels and Definition 4.3, it is clear that \( f \in \{f \in \mathcal{H}^n: \Lambda_j f = 0, j = 1, 2, \cdots, m\} \) if and only if \( f \in \bigcup_{j=1}^{m} \ker \Lambda_j \).

**Definition 4.4** Let \( \Lambda_j \in \mathcal{L}(\mathcal{H}^n, \mathcal{H}_j), j = 1, 2, \cdots, m. \) If the sequence \( \{\Lambda_j\}_{j=1}^{m} \) is g-complete and there exist \( A, B > 0 \) such that for every subset \( X \subset \{1, 2, \cdots, m\} \) and \( f_j \in \mathcal{H}_j \),

\[ A \sum_{j \in X} \|f_j\|^2 \leq \left\| \sum_{j \in X} \Lambda_j^* f_j \right\|^2 \leq B \sum_{j \in X} \|f_j\|^2. \]

Then \( \{\Lambda_j\}_{j=1}^{m} \) is called a g-Riesz basis for \( \mathcal{H}^n \) with respect to \( \{\mathcal{H}_j\}_{j=1}^{m} \).
Lemma 4.3: If \( \mathcal{H}_j = \mathbb{R}, \forall j = 1,2, \cdots, m \), there exists a frame \( \{f_j\}_{j=1}^m \) for \( \mathcal{H}^n \) such that \( \Lambda_j^* x = x f_j, \forall x \in \mathbb{R} \). Since \( \{\Lambda_j\}_{j=1}^m \) is a g-Riesz basis for \( \mathcal{H}^n \) with respect to \( \{h_j\}_{j=1}^m \), we have

\[
A \sum_{j \in X} \|x_j\|^2 \leq \left\| \sum_{j \in X} \Lambda_j^* x_j \right\|^2 = \left\| \sum_{j \in X} x_j f_j \right\|^2 \leq B \sum_{j \in X} \|x_j\|^2, \quad \forall X \subseteq \{1,2, \cdots, m\},
\]

which means \( \{f_j\}_{j=1}^m \) is a Riesz basis for \( \mathcal{H}^n \).

After given the definitions above, we start to study characterizations of g-frame and g-Riesz basis. Let \( \Lambda_j \in \mathcal{L}(\mathcal{H}^n, \mathcal{H}_j) \). Suppose that \( \{e_{j,k}\} \) is an orthonormal basis for \( \mathcal{H}_j \). Then \( \Phi: \mathcal{H}^n \rightarrow \mathbb{R}, \quad f \mapsto \langle \Lambda_j f, e_{j,k} \rangle \) is a linear map. Consequently, there exists \( u_{j,k} \in \mathcal{H}^n \) such that

\[
\langle f, u_{j,k} \rangle = \langle \Lambda_j f, e_{j,k} \rangle, \quad \forall f \in \mathcal{H}^n.
\]

Let \( \text{dim } \mathcal{H}_j = n_j \), we have

\[
\Lambda_j f = \sum_{k=1}^{n_j} \langle f, u_{j,k} \rangle e_{j,k}, \quad \forall f \in \mathcal{H}^n.
\]

Thus, we have

\[
\sum_{k=1}^{n_j} |\langle f, u_{j,k} \rangle|^2 = \left\| \sum_{k=1}^{n_j} \langle f, u_{j,k} \rangle e_{j,k} \right\|^2 = \|\Lambda_j f\|^2 \leq \|\Lambda_j\|^2 \|f\|^2.
\]

Therefore, \( \{u_{j,k}\}_{k=1}^{n_j} \) is a Bessel sequence for \( \mathcal{H}^n \). Then

\[
\langle f, \Lambda_j^* g \rangle = \langle \Lambda_j f, g \rangle = \sum_{k=1}^{n_j} \langle f, u_{j,k} \rangle \langle e_{j,k}, g \rangle = \left( f, \sum_{k=1}^{n_j} \langle e_{j,k}, g \rangle u_{j,k} \right), \forall f \in \mathcal{H}^n, g \in \mathcal{H}_j.
\]

Therefore, we have

\[
\Lambda_j^* g = \sum_{k=1}^{n_j} \langle e_{j,k}, g \rangle u_{j,k}, \quad \forall g \in \mathcal{H}_j.
\]

Therefore, \( u_{j,k} = \Lambda_j^* e_{j,k} \). In this case, \( \{u_{j,k}: 1 \leq j \leq m, \ 1 \leq k \leq n_j\} \) is called the sequence induced by \( \{\Lambda_j\}_{j=1}^m \) with respect to \( \{e_{j,k}: 1 \leq j \leq m, \ 1 \leq k \leq n_j\} \).

Now we start to study the g-frame and g-Riesz basis using \( \{u_{j,k}: 1 \leq j \leq m, \ 1 \leq k \leq n_j\} \). The following theorem would give a necessary and sufficient condition that the sequence \( \{\Lambda_j\}_{j=1}^m \) is a g-frame (respectively g-Riesz basis) for \( \mathcal{H}^n \).

**Theorem 4.1** Let \( \Lambda_j \in \mathcal{L}(\mathcal{H}^n, \mathcal{H}_j) \) and \( u_{j,k} \) be the sequence induced by \( \{\Lambda_j\}_{j=1}^m \) with respect to \( \{e_{j,k}: 1 \leq j \leq m, \ 1 \leq k \leq n_j\} \). Then, \( \{\Lambda_j\}_{j=1}^m \) is a g-frame (respectively g-Riesz basis) for \( \mathcal{H}^n \) if and only if \( \{u_{j,k}: 1 \leq j \leq m, \ 1 \leq k \leq n_j\} \) is a frame (respectively Riesz basis) for \( \mathcal{H}^n \).

**Proof.** Since \( \Lambda_j f = \sum_{k=1}^{n_j} \langle f, u_{j,k} \rangle e_{j,k} \) from above, we have

\[
\sum_{j=1}^{m} \|\Lambda_j f\|^2 = \sum_{j=1}^{m} \left\| \sum_{k=1}^{n_j} \langle f, u_{j,k} \rangle e_{j,k} \right\|^2 \leq \sum_{j=1}^{m} \sum_{k=1}^{n_j} |\langle f, u_{j,k} \rangle|^2.
\]

Thus, \( \{\Lambda_j\}_{j=1}^m \) is a g-frame for \( \mathcal{H}^n \) if and only if \( \{u_{j,k}\}_{1 \leq j \leq m, 1 \leq k \leq n_j} \) is a frame for \( \mathcal{H}^n \).
Now we turn to prove the part of g-Riesz basis. \( \{e_{j,k}\} \) is an orthonormal basis for \( \mathcal{H}_j \), therefore, for every \( g_j \in \mathcal{H}_j \), there exist \( \{a_{j,k}\} \subset K \) such that \( g_j = \sum_{k=1}^{n_j} a_{j,k} e_{j,k} \). Assume that \( \{A_j\}_{j=1}^m \) is a g-Riesz basis for \( \mathcal{H}^n \). Then, there exist \( A, B > 0 \) such that

\[
A \sum_{j \in \mathcal{X}} \|g_j\|^2 \leq \left\| \sum_{j \in \mathcal{X}} A_j^* g_j \right\|^2 \leq B \sum_{j \in \mathcal{X}} \|g_j\|^2.
\]

Thus,

\[
A \sum_{j \in \mathcal{X}} \sum_{k=1}^{n_j} |a_{j,k}|^2 \leq \left\| \sum_{j \in \mathcal{X}} \sum_{k=1}^{n_j} a_{j,k} u_{j,k} \right\|^2 \leq B \sum_{j \in \mathcal{X}} \sum_{k=1}^{n_j} |a_{j,k}|^2.
\]

Therefore, \( \{u_{j,k} : 1 \leq j \leq m, 1 \leq k \leq n_j\} \) is a Riesz basis for \( \mathcal{H}_j \).

Now we prove the necessity. Since \( \Lambda_j f = \sum_{k=1}^{n_j} \langle f, u_{j,k} \rangle e_{j,k} \), the set \( \{f \in \mathcal{H}^n : \Lambda_j f = 0, j = 1, 2, \ldots, m\} = \{f \in \mathcal{H}^n : \langle f, u_{j,k} \rangle = 0, \forall 1 \leq j \leq m, 1 \leq k \leq n_j\} \). Therefore, \( \{A_j\}_{j=1}^m \) is g-complete if and only if \( \{u_{j,k} : 1 \leq j \leq m, 1 \leq k \leq n_j\} \) is complete in \( \mathcal{H}_j \). Thus, \( \{A_j\}_{j=1}^m \) is a g-Riesz basis for \( \mathcal{H}_j \) if and only if \( \{u_{j,k} : 1 \leq j \leq m, 1 \leq k \leq n_j\} \) is a Riesz basis for \( \mathcal{H}^n \).

5. Applications

5.1 Frames Algorithm and Constructions

5.1.1 Signal Recovery by Frames

In this section, we would like to introduce how to construct a frame. Let \( \{f_i\}_{i=1}^m \) be a frame for \( \mathcal{H}^n \) with frame bound \( A \), \( B \) and frame operator \( S \). Then the signal \( x \) can be reconstructed as

\[
x = \sum_{i=1}^{m} \langle x, f_i \rangle S^{-1} f_i.
\]

However, this method is usually expensive and difficult to put into application. Thus, we give a more accurate or less expensive method, i.e. frame algorithms.

**Proposition 5.1** (Frame Algorithms, see in [5][8]) Let \( \{f_i\}_{i=1}^m \) be a frame for \( \mathcal{H}^n \) with frame bound \( A \), \( B \) and frame operator \( S \). Given a signal \( x \), we define the sequence \( \{y_n\}_{n=0}^\infty \) by \( y_0 = 0 \) and

\[
y_n = y_{n-1} + 2 \frac{B}{A+B} (S(x - y_{n-1}) - S(y_{n-1})), \quad \forall n \geq 1.
\]

Here, \( a_n = \langle x, f_i \rangle \), which is only related to the sample set. \( y_n \) is convergent to \( x \) in \( \mathcal{H}^n \) and the rate of the convergence is

\[
\|x - y_n\| \leq \left( \frac{B-A}{B+A} \right)^n \|x\|.
\]

This is a classic solution to recover a signal. But its convergence speed is closely related to the bound of the frame. When the bound ratio of the frame is larger than the column, the convergence speed of the sequence \( \{y_n\}_{n=0}^\infty \) is slow. Chebyshev Algorithm and Conjugate Gradient Method would solve this problem, see in [8].
5.1.2 Frame Construction

Once we understand how to recover signals through frames, we would introduce how to construct a frame. We have different construction methods for different frames. We take the tight frame and p-frame as examples to introduce some methods.

Proposition 5.2 Let \( \{f_i\}_{i=1}^m \) be a frame for \( \mathcal{H}^n \) with frame operator \( S \). Then, \( \left\{ S^{-\frac{1}{2}} f_i \right\}_{i=1}^m \) is a Parseval frame.

For the more general tight frame, we have the following proposition.

Proposition 5.3 Let \( m \leq n \) be integer and \( a_1 \geq a_2 \geq \cdots \geq a_m \) be positive real numbers. The followings are equivalent.

1. there exists a tight frame \( \{f_i\}_{i=1}^m \) for \( \mathcal{H}^n \) such that \( \|f_i\| = a_i \), \( \forall \ i = 1, 2, \cdots, m \).
2. \( a_i^2 \leq \frac{\sum_{j=i+1}^m a_j^2}{n-i} \), \( \forall \ i = 1, 2, \cdots, n - 1 \).
3. \( \sum_{i=1}^m a_i^2 \geq na_1^2 \).

Proof. See [4].

The methods above involve how to get one frame from another, which leads to another important topic, how to erasure some vectors so that the remaining vectors still form a frame. A frame \( \{f_i\}_{i=1}^m \) for \( \mathcal{H}^n \) is a generic frame if the erasure of any \( m-n \) vectors leaves a frame. How to determine whether a frame is a generic frame is difficult but meaningful. Because in practical applications, if a frame is a generic frame, we can greatly save costs, but we do not have a good way to judge it.

5.2 Dynamical Sampling

Dynamic sampling is a hot topic right now. The important point of this question is when coarse sampling and more detailed sampling have the same results. The key to it is how to find a condition so that the samples we collect can recover the original signal. From a mathematical point of view, we assume that the original signal \( f \) is a vector in \( l^2(I) \), where \( I = \{1, 2, \cdots, d\} \) and \( f \) is evolved by \( A \). We record the sample at time \( t \) and position \( i \) as \( A^i f(i) \). Thus, we get the set \( Y = \{f(i), Af(i), \cdots, A^i f(i) : i \in \Omega \subset I\} \) which is called the sample set. Therefore, the question is: under what circumstances would the sample points restore the original signal \( f \) within a certain error range? In other words, what conditions does \( A, \Omega \) satisfy so that the original signal \( f \) can be recovered in a stable way?

First, we introduce what is "in a stable way". Let \( Y = \{f(i), Af(i), \cdots, A^i f(i) : i \in \Omega \subset I\} \) be the sample set. Consider \( S_i : l^2(I) \to l^2([0,1,\cdots,l_i]) \), defined by \( S_i f = \left(A^j f(i)\right)_{j=0,1,\cdots,l_i} \).

Let \( S = \bigoplus_{i \in \Omega} S_i \). Then we have the following definition.

Definition 5.1 Let \( f \) be a vector in \( l^2(I) \) and \( K \in \mathcal{L}(l^2(I)) \). \( f \) can be recovered from the sample set \( Y \) in a stable way with respect to \( K \) if and only if there exist constants \( c_1, c_2 > 0 \) such that
\[
c_1 \|K^* f\|_2^2 \leq \|S f\|_2^2 = \sum_{i \in \Omega} \|S_i f\|_2^2 \leq c_2 \|f\|_2^2
\]
If \( K = id \), then we have
\[
c_1 \|f\|_2^2 \leq \|S f\|_2^2 = \sum_{i \in \Omega} \|S_i f\|_2^2 \leq c_2 \|f\|_2^2,
\]
and we call that \( f \) can be recovered from the sample set \( Y \) in a stable way.
Lemma 5.1 Let \( \{e_i\} \) be the standard basis for \( l^2(I) \). Then, every \( f \in l^2(I) \) can be recovered from the sample set \( Y = \{ f(i), Af(i), \ldots, A^lf(i) : i \in \Omega \subset I \} \) in a stable way with respect to \( K \) if and only if the set of vectors \( \{A^je_i : i \in \Omega, j = 0, \ldots, l_1\} \) is a \( K \)-frame for \( l^2(I) \).

Proof. Since \( A^nf(i) = \langle A^nf, e_1 \rangle = \langle f, A^*n e_1 \rangle \) and \( \|S_i f\|_2^2 = \sum_{j=0}^{l_1} |\langle f, A^j e_1 \rangle|^2 \), we would obtain from the definition that

\[
c_1 \|K^*f\|_2^2 \leq \sum_{i \in \Omega} \sum_{j=0}^{l_1} |\langle f, A^j e_1 \rangle|^2 \leq c_2 \|f\|_2^2.
\]

Thus, the set of vectors \( \{A^je_i : i \in \Omega, j = 0, \ldots, l_1\} \) is a \( K \)-frame for \( l^2(I) \).

Remark. If \( K = Id \), we have every \( f \in l^2(I) \) can be recovered from the sample set \( Y = \{ f(i), Af(i), \ldots, A^lf(i) : i \in \Omega \subset I \} \) in a stable way if and only if the set of vectors \( \{A^je_i : i \in \Omega, j = 0, \ldots, l_1\} \) is a frame for \( l^2(I) \).

Although we write the sample set as \( \{ f(i), Af(i), \ldots, A^lf(i) : i \in \Omega \subset I \} \), the operator \( A \) is not easy to get directly. Therefore, we begin to show how to calculate the operator \( A \). In finite dimensional case, \( A \) would be regarded as a matrix. Let \( B \) be an invertible matrix. Let \( Q \) be the matrix that satisfies \( Q = BA^*B^{-1} \). Let \( b_i \) denote the \( i \)th column of \( B \). We have the following.

Lemma 5.2 Let \( f \) be a vector in \( l^2(I) \) and \( K \in L(l^2(I)) \). \( f \) can be recovered from the sample set \( Y = \{ f(i), Af(i), \ldots, A^lf(i) : i \in \Omega \subset I \} \) in a stable way with respect to \( K \) if and only if the set of vectors \( \{Q^jb_i : i \in \Omega \subset I, j = 0, 1, \ldots, l_1\} \) is a \( K \)-frame for \( l^2(I) \).

Proof. By Lemma 5.1, we have every \( f \in l^2(I) \) can be recovered from the sample set \( Y = \{ f(i), Af(i), \ldots, A^lf(i) : i \in \Omega \subset I \} \) in a stable way with respect to \( K \) if and only if the set of vectors \( \{A^je_i : i \in \Omega, j = 0, \ldots, l_1\} \) is a \( K \)-frame for \( l^2(I) \). Since a \( K \)-frame is transformed by invertible linear operators, \( \{A^je_i : i \in \Omega, j = 0, \ldots, l_1\} \) is a \( K \)-frame for \( l^2(I) \) if and only if \( \{Q^jb_i : i \in \Omega \subset I, j = 0, 1, \ldots, l_1\} \) is a \( K \)-frame for \( l^2(I) \).

This lemma means if we would like to find out what \( A \) is, we only need to calculate the Jordan normal form \( J \). It is well-known that

\[
J = \begin{pmatrix} J_1 & 0 & \cdots & 0 \\ 0 & J_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & J_n \end{pmatrix}, \quad \text{where} \quad J_i = \begin{pmatrix} \lambda_i & 1 & \cdots & 0 \\ 0 & \lambda_i & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_i \end{pmatrix}.
\]

Thus, we only need to study the eigenvalues and eigenvectors of \( A \). In specific practical applications, it is obviously easier for us to calculate the eigenvalues and eigenvectors of linear operators. Thus, we might better determine whether a signal \( f \) can be recovered.

5.3 Gabor Frames

In practical applications, another widely used frame is Gabor frame. We would introduce it in this section. First, we introduce the definitions of some necessary operators. Let \( \mathcal{H}^n \) be an n-dimensional Hilbert space. \( \{e_i\}_{i=1}^{n} \) is an orthonormal basis of \( \mathcal{H}^n \). Then, the following two operators are significant in Gabor frame. One of them is the circular shift operator \( C : \mathcal{H}^n \to \mathcal{H}^n \) given by

\[
C x = x(n)e_1 + x(1)e_2 + \cdots + x(n-1)e_n, \quad \forall x \in \mathcal{H}^n,
\]

where \( x = \sum_{k=1}^{n} x(k)e_k \). Then, the translation operator \( C_k \) by \( k \in \{0,1,\cdots,n-1\} \) is given by \( C_k(x) = C^K(x) \). The other one is modulation operator, which is given by \( M_l : \mathcal{H}^n \to \mathcal{H}^n \), \( l \in \{0,1,\cdots,n-1\} \) and

\[
M_l x = e^{-\frac{2\pi i}{n}} x(1)e_1 + e^{-\frac{2\pi i}{n}} x(2)e_2 + \cdots + e^{-\frac{2\pi i}{n}} x(n)e_n, \quad \forall x \in \mathcal{H}^n.
\]

Applying \( M_l \) and \( T_k \) we could introduce the definition of the time frequency shift operator. It is given by \( \pi(k,l) : \mathcal{H}^n \to \mathcal{H}^n \),

\[
\pi(k,l)x = M_l T_k x, \quad \forall x \in \mathcal{H}^n
\]

Then we can apply the time frequency shift operator to introduce the definition of the Gabor frame and Gabor \( K \)-frame.
Let \( \mathbf{v} \) be a non-zero vector in \( \mathcal{H}^n \) and \( \Lambda = \{1,2,\cdots,n\} \times \{1,2,\cdots,n\} \). Then we call \((\mathbf{v}, \Lambda) = \{\pi(k,l)\mathbf{v}\}_{(k,l)\in \Lambda}\) the Gabor system generated by the window function \( \mathbf{v} \) and \( \Lambda \). If a Gabor system \( \Lambda \) is a frame for \( \mathcal{H}^n \), we call it the Gabor frame for \( \mathcal{H}^n \). If a Gabor system \( \Lambda \) is a \( K \)-frame for \( \mathcal{H}^n \), we call it the Gabor \( K \)-frame for \( \mathcal{H}^n \).

6. Conclusion

In this article, we introduce the definition and properties of finite-dimensional frames and generalized frames. Based on these, we introduced how to construct a frame and how to recover a signal from the frames. Moreover, we take dynamic sampling as an example to introduce the application of the frames in practical problems, and propose possible research directions for future research on frames application.

References